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V.P. Ivaschenko, G.G. Shvachych, M.A. Tkach

SPECIFICS OF CONSTRUCTING OF MAXIMALLY PARALLEL ALGORITHMIC FORMS OF THE SOLVING OF THE APPLIED TASKS

The features of construction of maximally parallel algorithms of thermal conductivity equation's solution with the method of direct tasks of Dirichlet and Neumann are considered. Parallelization of the three-diagonal systems of equations allows to construct absolutely steady algorithms, having a maximal parallel form. It is achieved minimum time period of solving the applied tasks on parallel computing devices.

Keywords: parallel algorithms, net functions, method of lines, conception of discretization.

PROBLEM STATEMENT AND ANALYSIS OF THE LAST ARCHIEMENTS IN THIS AREA

The parallel computation systems develop quickly. When computer clusters came into service, parallel computations became available for many people. Mass processors, standard net technologies and free software are used as a rule to construct clusters. Let's note that the computer cluster is the set of processors united in the framework of a certain network to solve one problem. A small computer cluster with 6 processors may be used efficiently by small departments. In this connection, the problem of efficient software development becomes one of the central problems of parallel computations in the whole now. Creation of parallel computation systems demanded the mathematical conceptions development of parallel algorithm construction, that is, the algorithms adapted to realization on similar computation systems [1 – 7]. On the other hand the development of mathematical modeling leads to the more complex descriptions of the models. To understand them and to develop the analysis principles one has to involve the newest achievements from quite different mathematical fields. Problem discretization results in systems of linear equations with a great number of unknowns. Former methods of their solution are not always suitable from the point of view of accuracy, rate, required memory, algorithm structure and the like. The new ideas in the field of computational mathematics arise and are realized. In the final analysis, the new methods of numerical experiments realization are created for more perfect mathematical models [8 – 13]. In this work the efficacy of system of linear algebraic equations (SLAE) parallelization of three-diagonal structure using the numerical-and-analytical method of lines is shown in terms of the simplest heat conductivity problem solution.

MATHEMATICAL PROBLEM STATEMENT

Let's examine the Dirichlet boundary value problem solution for a one-dimensional heat conduction equation

$$\frac{\text{ä}\mathrm{Y}}{\text{ä}t} = a\frac{\text{ä}^2\mathrm{Y}}{\text{ä}\text{ő}^2}, \quad t \in [t_0, T], \, x \in [x_0, x_L] \tag{1}$$

with the initial condition

$$\mathrm{Y}\big|_{t=t_0} = \phi(\text{ő}) \tag{2}$$

and the boundary condition

$$\mathrm{Y}\big|_{\text{ő}=\text{ő}_0} = \mathrm{Y}W(t), \; Y\big|_{x=x_L} = \mathrm{Y}L\;(t), \tag{3}$$

Let's juxtapose a net domain to a definitional domain of sought function Y(t, x) in (1) – (3) problem

$$\left.\begin{aligned} t_j &= j \times Dt1,\; j = \overline{1, M},\; Dt1 = T / M,\; M \in Z \\ x_p &= p \times Dx1,\; p = \overline{0, 2m},\; Dx1 = (x_L - x_0)/2m,\; m \in Z \end{aligned}\right\}. \tag{4}$$

Let's introduce new independent variables to normalized units

$$\left.\begin{aligned} \varepsilon_t &= \frac{t - t_{j-1}}{t_j - t_{j-1}} \in [0,1] \\ \varepsilon_x &= \frac{x - x_p}{x_{p+1} - x_p} \in [-1,1] \end{aligned}\right\}. \tag{5}$$

here we'll get:

$$\frac{\partial Y_{p+\varepsilon_x,1}}{\partial \varepsilon_t} = A\frac{\partial^2 Y_{p+\varepsilon\;y,1}}{\partial \varepsilon_x^2}, \qquad A = Dt1\frac{a}{Dx1^2} \tag{6}$$

where $\mathrm{Y}_{p+\varepsilon x,1}(\varepsilon_t, \varepsilon_x)$ is sought piecewise-analytic function on spatial variable.

MAIN PART OF RESEARCH

Discretization scheme by method of lines.The conception of (1) – (3) problem discretization by method of lines lies in the following. After the equation (6) finite-difference approximation on a temporary variable

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we'll get the second-order system of ordinary differential equations (SODE).

$$Y^{//}_{p+\varepsilon_x,1}(\varepsilon_x)-\frac{1}{A}Y_{p+\varepsilon x,1}(\varepsilon_x)=-\frac{1}{A}YO_{p+\varepsilon x,1}(\varepsilon_x), \tag{7}$$

where $YO_{p+\varepsilon x,1}(\varepsilon_x)$ is known initial function.

General solution of equation (7) defines in the final form:

$$Y_{p+\varepsilon_{x,1}}(\varepsilon_x)=Y^{*}_{p+\varepsilon_{x,1}}(\varepsilon_x)+C_p C\eta\beta(\varepsilon_x)+D_p S\eta\beta(\varepsilon_x)\,, \tag{8}$$

where C_p, D_p are the constants of integration;

$Y_{P+\varepsilon x,1}(\varepsilon_x)$ – specific solution of heterogeneous equation (7);

$\beta=\sqrt{\frac{1}{A}}$ eigen-values of characteristic equation

$$\beta^2-\frac{1}{A}=\Delta\,. \tag{9}$$

With differentiating solution (8) on ε_x,, we get

$$Y_{p+\varepsilon_{x,2}}(\varepsilon_x)=Y^{*}_{p+\varepsilon_{x,2}}(\varepsilon_x)+\beta\left[C_p C\eta\beta(\varepsilon_x)+D_p S\eta\beta(\varepsilon_x)\right], \tag{10}$$

where $Y_{p+\varepsilon x,2}(\varepsilon_x)=Dx1\frac{\partial Y_{p+\varepsilon_{x,1}}}{\partial x}\Big|_{t=t_j}$ is an unknown gradient function.

Determine constants of integration in ratios (8), (10) from the conditions when

εx = ±1:

$$Y_{P+\varepsilon x,1(2)}(\varepsilon_x)\big|_{\varepsilon_x=\pm 1}=Y_{p\pm 1,1(2)}. \tag{11}$$

get

$$Y_{\varsigma+\varepsilon x,1}(\varepsilon_x)=\left\{\begin{aligned}&Y^{*}_{\varsigma+\varepsilon_{x,1}}(\varepsilon_x)+\frac{S\eta\beta(1+\varepsilon_x)}{S\eta\beta(2)}\left[Y_{p+1,1}-Y^{*}_{p+1,1}\right]+\\&+\frac{S\eta\beta(1-\varepsilon_x)}{S\eta\beta(2)}\left[Y_{p-1,1}-Y^{*}_{p-1,1}\right]\end{aligned}\right\} \tag{12}$$

$$Y_{3+\varepsilon x,2}(\varepsilon_x)=\left\{\begin{aligned}&Y^{*}_{3+\varepsilon_{x,2}}(\varepsilon_x)+\frac{S\eta\beta(1+\varepsilon_x)}{S\eta\beta(2)}\left[Y_{p+1,2}-Y^{*}_{p+1,2}\right]+\\&+\frac{S\eta\beta(1-\varepsilon_x)}{S\eta\beta(2)}\left[Y_{p-1,2}-Y^{*}_{p-1,2}\right]\end{aligned}\right\} \tag{13}$$

Having put εx = 0, let's transform solution (12), (13) to their discrete analogues in the form of SLAE:

$$C_p \begin{Bmatrix} Y_{p+1,1} \\ Y_{p+1,2} \end{Bmatrix} - \begin{Bmatrix} Y_{p,1} \\ Y_{p,2} \end{Bmatrix} + D_p \begin{Bmatrix} Y_{p-1,1} \\ Y_{p-1,2} \end{Bmatrix} = \begin{Bmatrix} f_{p,1} \\ f_{p,2} \end{Bmatrix} \tag{14}$$

where

$$\left. \begin{aligned} & C_p = \frac{S\eta\beta(1)}{S\eta\beta(2)}, \qquad D_p = \frac{Su\beta(1)}{Su\beta(2)} \\ & \begin{Bmatrix} f_{p,1} \\ f_{p,2} \end{Bmatrix} = C_p \begin{Bmatrix} Y^*_{p+1,1} \\ Y^*_{p+1,2} \end{Bmatrix} - Y^*_{p,1(2)} + D_p \begin{Bmatrix} Y^*_{p,1} \\ Y^*_{p,2} \end{Bmatrix} \end{aligned} \right\}, \quad p = \overline{1, 2m-1}. \tag{15}$$

SLAEs (14) are invariant relative to net functions Yp,1 and Yp,2 and have three-diagonal structure where boundary elements Y0,1, Y2m,1 for the problem (1)–(3) are identical to the values of boundary functions (3):

$$Y_{0,1} = YW(t_j), \quad Y_{2m,1} = WL(t_j) \tag{16}$$

Identically, Y0,2, Y2m,2 elements on the domain boundary in SLAE (14) take the following values:

$$\left. \begin{aligned} & Y_{0,2} = Dx1 \times \frac{\partial Y}{\partial x}\Big|_{\substack{x=x_0 \\ t=t_{oj}}} \\ & Y_{2m,2} = Dx1 \times \frac{\partial Y}{\partial x}\Big|_{\substack{x=x_L \\ t=t_j}} \end{aligned} \right\}, \tag{17}$$

which in Neumann's problems correspond to second-order boundary data.

Thus, the mathematical model invariance in the form of SLAE (14) relative to Yp, 1, Yp, 2 net functions apparently reflects deeper group-theoretical features of input equation (1) relative to Coshi's data as well. In light of noted, the differential manifold and Dirichlet's and Neumann's problems group classification being carried out on Coshi's data in SLAE permits to continue the analysis on the pattern of only one group of decision variables Yp, 1 examination.

SWEEP METHOD ANALYSIS

Let's describe a simple and ordinary solving method of SLAE (14) named the sweep method. Let's postulate the existence of such two vectors E and G that for whatever

$Y_{p,1}$ $(p = \overline{1, 2m-1})$ the following equality is executed:

$$Y_{p,1} = E_p Y_{p+1,1} + G_p \tag{18}$$

Having reduced into (18) index p by a unit, we get

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$$\mathrm{Y}_{p-1,1} = E_{p-1}\mathrm{Y}_{p,1} + G_{p-1} \tag{19}$$

After ratio (19) substitution to SLAE (14) we'll find

$$\mathrm{Y}_{p,1} = \frac{C_p}{1-D_pE_{p-1}}\mathrm{Y}_{p+1,1} + \frac{D_pG_{p-1} - f_{p,1}}{1-D_pE_{p-1}} \tag{20}$$

When comparing equations (18) and (20) and noting that both of the equations being equitable for all the $p = \overline{1,2m-1}$ indices get the recurrent ratios:

$$E_p = \frac{C_p}{1-D_pE_{p-1}}, \quad G_p = \frac{D_pG_{p-1} - f_{p,1}}{1-D_pE_{p-1}} \tag{21}$$

realizing the algorithm of direct sweep. In fact, the following goes from the conditions on the left boundary (3):

$$E_0 = 0, \quad G_0 = \mathrm{Y}_{0,1} = YW(t_j) \tag{22}$$

Further Ep, Gp elements are calculated in all the points in increment direction $p = \overline{1,2m-1}$ on the recurrent formulae (21). Then it goes from the right boundary condition (3) that $\mathrm{Y}_{2m,1} = \mathrm{Y}L(t_j)$. This provides for start of reverse sweep on the recurrent formula (19) in decrease p from p=2m-1 up =1.

If a sweep algorithm on formulae (20), (18) has the right orientation, it is obvious that it is possible to organize a sweep algorithm of the opposite left orientation as well. Let's assume the existence of such E and G vectors that for all the Yp,1 the following equality is executed:

$$\mathrm{Y}_{p,1} = E_p\mathrm{Y}_{p-1,1} + G_p \tag{22}$$

Having changed in (22) index p by increasing per a unit, we get

$$\mathrm{Y}_{p+1,1} = E_{p+1}\mathrm{Y}_{p,1} + G_{p+1}. \tag{23}$$

Then after elimination from SLAE (14) Yp+1, 1 variables by substitution on formulae (23), it seems possible to develop the following recurrent dependencies:

$$E_p = \frac{D_p}{1-C_pE_{p+1}}, \quad G_p = \frac{C_pG_{p+1} - f_{p,1}}{1-C_pE_{p-1}} \tag{24}$$

corresponding to the right sweep algorithm in index p from p=2m-1 up p=1 decrease direction. Here the reverse sweep is realized on the recurrent ratios (22) in increment direction of p = 1.2m -1 index, which was to be proved.

Let's note that to calculate one using sweep method the system solution (14), consisting of (2m+1) equations, it's necessary to carry through arithmetical operations in the quantity being only by finite number times bigger than unknown quantity. To solve equations with N unknown quantities of arbitrary linear system N using method of exclusion one usually has to use up arithmetic operations in N3 quantity. They managed to attain such a reduction of arithmetic operations number when system (14) solving by sweep method while successfully using the specific character of this system.

CONCLUSION

Three-diagonal SLAE parallelization based on numerical-and-analytical method of lines allows to construct absolutely steady algorithms, having a maximal parallel form and, thereby, to achieve minimum time period of its implementation on parallel computing devices. Noteworthy it is also the fact that input data computing errors separated from round-off ones inherent in real personal electronic computers. The algorithm is illustrated on the pattern of initial-boundary-value Dirichlet problem solution for thermal conduction equation. However, it turns out to be an invariant one for class of problems of Neumann. That's why all the input functions are replaced by input data of Naumann's proble $Y_{p,2}, Y^{*}_{p+\varepsilon_x,2}(\varepsilon_x)$ in the solving of Dirichlet problem. This fact obviously also reflects deeper group-theoretic properties of parabolic type equations relatively to Coshi's data.

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