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MODAL SYNTHESIS OF COMPENSATED OPTIMAL SYSTEMS WITH DELAY

Annotation. *This article proposes modal synthesis for linear stationary closed loop systems with the lag using optimal control law in the form of a linear combination of the state variables to ensure the specified dynamic characteristics. Procedure of modal synthesis of the optimal control law is performedbased onundefined coefficients method. The delay compensation method is proposed to use in order to eliminate the stable self-oscillations occurrence at the stabilization process near given trajectory.*

Keywords: *linear stationary system, stabilization mode, transport lag, modal synthesis, optimal law.*

Introduction

Systems synthesis task is one of the key tasks of both automatic control theory and practice. Its solution results in definition of the structure of the automatic control system and its parameters of the condition of the system sustainability and quality of transient processes (achieving the required performance, the inadmissibility of the considerable overshoot) improving control accuracy in steady-state conditions etc.Linear controllers are an effective way to ensure dynamic performance not only linear control objects of arbitrarily high order, but also objects that contain non-linear and discrete units which have a significant, but not a determining influence on dynamic processes.There are two main deterministic approaches to create the control system for the object's state vector - analytical design of optimal controllers and modal control.

Professor A.M. Letov [1,3] published his work in 1960, in which was obtained the analytical solution of the problem of linear stationary object's optimal stabilization with a quadratic quality functional, it was later called "analytic construction of regulators" (ACOR). Problem of linear non-stationary objects optimization was solved in Kalman's work [2] published in 1960.ACOR has the ultimate goal of obtaining control law purely analytically, based on the requirements for management quality [2]. Synthesis of the desired optimal closed loop control system using ACOR depends on the designer choice of suitable penalty matrix coefficient values to obtain a minimum of quality criterion is not quite convenient because of absence of obvious relationship between selected coefficients and transients in a closed-loop system.In addition, the application of the ACOR method leads to the necessity of solving nonlinear matrix Riccati equation, which is a non-trivial task and requires the use of special numerical procedures.

The essence of the modal synthesis of optimal control is to determine the numerical values of the delayless feedback transmission coefficients in all the variables of the object state in order to ensure a predetermined distribution of the characteristic equation roots (eigenvalues) in the closed-loop control system. The majority of industrial objects have delays. The delay may occur due to time spent on signal transmission or, as happens more often can be caused by the phenomenon of simplifying assumptions, by virtue of which it is considered that action of intermediate and reinforcing links in the controlled object is reduced to a signal transmission with delay [5,6]. Of particular interest is the training of remote control operators for various types of technical objects. Inertia of the operator himself has a significant impact on the management quality in addition to the delay in the signal transmission. Therefore, it's imperative to have the best (reference) dynamic realization (control laws) in preparation of the operator considering the inertia and delay in the control loop. In thisarticle authors proposed a procedure for the synthesis of the optimal modal stabilization of linear stationary systems with delay based on the method of undetermined coefficients with the simultaneous compensation of the delay to eliminate self-oscillations. This article is an extension of the work of the authors [4].

Method of undetermined coefficients

It is known [3] that for completely measurable systems of the form:

$$\begin{gathered}\dot{\bar{x}} = A\bar{x} + B\bar{u}, \\ \bar{y} = C\bar{x}.\end{gathered} \tag{1}$$

In the case of a quadratic quality criterion, extreme control is a linear function of state variables:

$$\bar{u} = \bar{p}^T \bar{x}. \tag{2}$$

Moreover, if the vector of feedback coefficients$\bar{p}$is chosen in such a way that the poles of the closed system (1) are located at preassigned arbitrary points, then the required dynamic properties will be provided in the closed system [4]. Thus, this problem is reduced to the choice of the optimal location of the poles and determination of the feedback coefficients.

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We show that the unknown coefficients $\overline{p_i}(i = \overline{1,n})$ of the characteristic determinant of a closed optimal system [7]:

$$det(\lambda) = |A + Bp^{-T} - I|\lambda =$$

$$\begin{vmatrix} a_{11} + b_1p_1 - \lambda & \dots & a_{1j}p_j & \dots & a_{1n} + b_1p_n \\ a_{j1} + b_jp_1 & \dots & a_{jj} + b_jp_j - \lambda & \dots & a_{jn} + b_jp_n \\ a_{n1} + b_np_1 & \dots & a_{nj} + b_np_j & \dots & a_{nn} + b_np_n - \lambda \end{vmatrix},$$

linearly enter into the expression for the coefficients of the characteristic polynomial of a closed system.

Indeed, let $\exists b_k/b_k \neq 0$. Then, subtracting the kth $(j \neq k)$ line from the jth line, multiplied byb_j/b_k, we get a determinant equal to the original, in which the feedback coefficients $\overline{p_i}(i = \overline{1,n})$ enter the kth line. Expanding it along this line and grouping the terms with the corresponding λ powers, we finally arrive at the following expression of the characteristic polynomial of the closed system (3) or (4):

$$H(\lambda) = \lambda^n + \left(\sum_{i=1}^{n} c_{n-1,i}p_i + d_{n-1}\right)\lambda^{n-1} + \dots + \left(\sum_{i=1}^{n} c_{0,i}p_i + d_0\right), \quad (3)$$

$$H(\lambda) = \lambda^n + \left(\overline{c}_{n-1}^{T}\overline{p} + d_{n-1}\right)\lambda^{n-1} + \dots + \left(\overline{c}_0^{T}\overline{p} + d_0\right). \quad (4)$$

We define the unknown parameters c_{ji}, d_j $(j = \overline{0, n-1}; i = \overline{1,n})$ in $n+1$ step using the undetermined coefficients method. To do this, we put $p_i = 0$ $(i = \overline{1,n})$ in the characteristic determinant at the first step and reveal it by one of the known numerical methods and find that the coefficients found for different powers of λ determine the unknown coefficients $d_j(j = \overline{0, n-1})$ in the expressions for the characteristic polynomial of the closed system for the corresponding powers of λ. In the next n steps, setting sequentially one of the coefficients $p_i(i = \overline{1,n})$ equal to one while others remain zero and revealing the characteristic determinant, we obtain expressions for the unknown parameter c_{ji} for the corresponding power $\lambda^j(j = \overline{0, n-1})$ in the characteristic polynomial of the closed system.

$$c_{ji} = f_i - d_i, \quad (5)$$

where f_i is the coefficient of the ith open characteristic determinant. On the other hand, the characteristic polynomial of a closed system with the desired roots $\lambda_1, \lambda_2, \dots, \lambda_n$ has the form [4]

$$F(\lambda)=\prod_{i=1}^{n}(\lambda-\lambda_i)=\sum_{j=0}^{n-1} l_j\lambda^j+\lambda^n. \qquad (6)$$

As a result, to determine the feedback coefficients $\bar{p}$ in expression (2), we equate the expressions for the coefficients for the same powers in (4) and (6) and obtain a system of linear algebraic equations:

$$\mathrm{col}\left(\bar{c}^T, \bar{c}^T, \ldots, \bar{c}^T\right)\bar{p} = \bar{l} - \bar{d}, \qquad (7)$$

where $\bar{l} = (l_{n-1}, l_{n-1}, \ldots, l_0), \bar{d} = (d_{n-1}, d_{n-2}, \ldots, d_0).$

Now consider the procedure for modal synthesis based on the undefined coefficients method proposed for linear dynamical systems with transport delay[5,6].

Formulation of the problem

The dynamics model of the object is described as

$$\dot{\bar{x}} = A\bar{x} + By, \qquad (8)$$

where $\bar{x} = (x_1, x_2, \ldots, x_n)^T$ is fully measured vector of system states deviation from a predetermined trajectory of movement; A, B - coefficient matrix with dimension $n \times n$, $n \times l$; y - a scalar, characterized by deviation of controls, taking into account the reaction of the operator, the dynamic model has the form

$$\dot{y} = \lambda_y y + d_u u(t-\theta), \qquad (9)$$

where λ_y, d_u, θ- constants determined by psychophysical features of operators(and besides $\lambda_y = -\frac{1}{T}; d_u = \frac{k}{T}$); $u(t)$ − scalar control action, which will be sought in the form (2). The objective is to determine the coefficients $\bar{p} = (p_1, p_2, \ldots p_n)^T$, providing some predetermined dynamic characteristics of the stabilization process and achieving sustainable programmed movement of the system (8).

Resolving stabilization problem

As the operator delayθ is sufficiently small value, we'll write the equation (9) as a

$$\dot{y}(t) = \lambda_y y(t) + d_u u(t) - d_u\theta\dot{u}(t). \qquad (10)$$

In that case, if in some way estimate or measure the condition of the operator $y(t)$, the system (8), (10) is fully observed and the problem is solved as follows.

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We introduce into consideration the advanced phase vector. Then the closing equation has the form $\tilde{x} = (x_1, x_2, \dots, x_n, x_{n+1} = y)^T$.

$$u = p^T \tilde{x} \quad (11)$$

and the characteristic polynomial of the closed-loop system (8), (10) takes the form:

$$\det\left(A^* - I\lambda\right) = \begin{vmatrix} A - I\lambda & B \\ -\dfrac{d_u\left(\bar{p}^T - \theta\bar{p}^T A\right)}{1 + d_u\theta p_{n+1}} & \dfrac{\lambda_y + d_u p_{n+1} - \theta\bar{p}^T B}{1 + d_n\theta p_{n+1}} - \lambda \end{vmatrix} = 0, \quad (12)$$

where A^* is a matrix $(n+1) \times (n+1)$, $\bar{p} = (p_1, p_2, \dots p_n)^T$.

It is known that the multiplication of all the elements of a row or column by the factor λ is equivalent to multiplying the determinant on λ[7]. Hence, the determinant (12) can be written

$$\det\left(A^* - I\lambda\right) = \frac{1}{1 + d_u\theta p_{n+1}} \begin{vmatrix} A - I\lambda & B \\ \bar{p}^T - \theta\bar{p}^T A & \lambda_y + d_u p_{n+1} - d_u\bar{p}^T\theta B - \lambda(1 + d_u\theta p_{n+1}) \end{vmatrix}$$

and therefore assuming that the $(1 + d_u\theta p_{n+1})^{-1} \neq 0$,
we'll put

$$\begin{vmatrix} A - I\lambda & B \\ \bar{p}^T - \theta\bar{p}^T A & \lambda_y + d_u p_{n+1} - d_u\theta\bar{p}^T B - \lambda(1 + d_u\theta p_{n+1}) \end{vmatrix} = 0. \quad (13)$$

It is easy to show that the determinant (13) is a polynomial of degree $(n+1)$ on λ, and its coefficients are linearly dependent on $\bar{p} = (p_1, p_2, \dots p_n, p_{n+1})^T$, i.e.

$$\det\left(A^* - I\lambda\right) = H\left(\lambda, \tilde{p}\right) = \lambda^{n+1} + \left(\tilde{d}_n^T \tilde{p} + d_n^0\right)\lambda^n + \dots + \left(\tilde{d}_n^T \tilde{p} + d_n^0\right) = 0 \quad (14)$$

Indeed, when uncovering the determinant (13) in the last line, in which each element is a linear combination of the coefficients p, we're getting the expression (14).

Determination of unknown coefficients $d_i, d_i^0 (i = \overline{0, n})$ *is* made similarly to the procedure cited in this paper above.When equating between the coefficients of the polynomial powers (14) and the polynomial with spectrum $\{\lambda_i\} (i = \overline{1, n+1})$ selected to provide specified quality parameters of transient processes

$$L(\lambda)=\prod_{i+1}^{k+1}(\lambda-\lambda_i)=\sum_{k=0}^{n+1}l_k\lambda^k, \qquad (15)$$

where $l_{n+1} = 1$, we get the joint system of linear algebraic equations

$$D_{n+1}p = \tilde{l}, \qquad (16)$$

where $D_{n+1}$is matrix with $(n + 1) \times (n + 1)$coefficients and $p\tilde{l}$ are column vectors with dimension $(n + 1)$. The solution of system **(16)** provides the defined spectrum $\{\lambda_i\}(i = \overline{1, n + 1})$ to closed-loop system. Frequently it is not possible to evaluate or measure the state of the operator $y(t)$ in real conditions. Then it is necessary to put $p_n + 1 \equiv 0$ in the closing equation **(11)**. As a result, the characteristic determinant of a closed-loop system has the form

$$\det\left(A^* - I\lambda\right) = \left|\frac{A - I\lambda}{d_u(\overline{p}^T - \theta\overline{p}^T A)}\middle|\frac{B}{\lambda_y - d_u\theta\overline{p}^T B - \lambda}\right|. \qquad (17)$$

Desired characteristic polynomial is determined, as in the previous case, bythe expression **(15)**. When equating the coefficients of the polynomials **(17)** and **(15)** with the same powers λ we obtain incompatible systems of linear algebraic equations in contrast to **(16)**

$$D_n\bar{p} = \tilde{l}, \qquad (18)$$

where D_n is matrix of $(n \times n)$ coefficients and $\bar{p}$is n-dimensional column vector. It is possible to use the least squares method [8], for solving such a system, according to which the vector of unknown coefficients $\bar{p}$ is approximately defined as

$$\bar{p} = (D_n^T D_n)^{-1} D_n \tilde{l}. \qquad (19)$$

Ensuring the absence of auto-oscillations near a given trajectory of the system

The optimal stabilization law **(11)** of the system **(8)**, synthesized, proposed by the method of indefinite coefficients, provides the given dynamic properties of the process of stabilization of the system in the event of deviations from the given (software) trajectory of motion. However, this law does not eliminate the occurrence due to the presence of a lag of stable self-oscillations at the end point of the stabilization process near the given trajectory of motion. To compensate for the delay, a modified Bess's method [8] is proposed, the essence of which is as follows.

Let the equation of optimal (on an arbitrary criterion) switching surface in the absence of lag in the system is known and has the form

$$\Phi(x_1, x_2, ..., x_n) = 0, \ \bar{x} \subset X^n. \tag{20}$$

Assuming that the function F is solved for one of n of its arguments, for example, x_1, we write (29) in the

$$x_1 + \varphi(x_2, x_3, ..., x_n) = 0; \ \bar{x} \subset X^n. \tag{21}$$

It should be noted that the condition of solvability is not necessarily taken for clarity. The surface of the switching (21) is shown in Fig.1, where ABC is some optimal trajectory of the forced motion of the system to the switching surface (21).Let for simplicity in the control circuit there is a scalar control influence with delay θ. In order that the trajectory of the ABC in this case remains, as in the system without delay, it is obvious that the optimum surface is the geometric point of the points from which, after a while time θ, under the forced motion of the system, the point depicting moves to the surface (21).The equation of the optimum surface of the switching of the compensated system in this case has the form

$$\Phi^*(x_1, x_2, ..., x_n) = 0, \ \bar{x} \subset X^n$$

or

$$x_1 + \varphi^*(x_{2,} x_3, ..., x_n) = 0, \ \bar{x} \subset X^n. \tag{22}$$

Denote the distance between the projections of the points B and C on the $x_1 \dots x_n$ axes $\Delta_{x_1} \dots \Delta_{x_n}$ respectively. Obviously, $\Delta_{x_i} (i = \overline{1, n})$ there are functions of the delay time θ, and the values $x_i = x_i + \Delta_{x_i} (i = \overline{1, n})$ are the current values of the coordinates, which represent the value of the coordinates of the system through the delay time θ.

For geometric reasons we have

$$x_1 + \varphi^*(x_2, ..., x_n, \theta) = \Delta_{x_1} - \varphi(x_2 + \Delta_{x_2}, ..., x_n + \Delta_{x_n})$$

or

$$\Phi^*(x_1, ..., x_n) = \Phi(x_1 + \Delta_{x_1}, ..., x_n + \Delta_{x_n}). \tag{23}$$

E*quation* (23) is a general equation for determining the function Φ^* of a given function Φ. To do this, according to Bess's method, it is enough to determine the values $\Delta_{x_i}(\theta) (i = \overline{1, n})$, and then substitute the future values of the system's coordinates $x_i + \Delta_{x_i}$. In the system equation of the system's switching without delay. However, the definition of these quantities is substantially related to the time interval between two sequential control switchings that is specific to the class of tasks considered and absent in the Bess [8] method, which is general in nature. To determine these values, we write the general solution for coordinates of the vector $\bar{x}$ in the "reverse" time $z = t_k - t$

$$x_i(z) = f_i(\overline{x}^0, u, z); \overline{x}^0 = (x_1(t_0), x_2(t_0), ..., x_n(t_0)), \qquad (24)$$

where u - controlling influence; $\bar{x}^0$ - vector of initial values of corresponding coordinates lying on the optimal surface of switching system without delay.

In the general case, when solving the optimal control tasks for the speed and fuel consumption, the system usually takes several switching operations. In this case, the importance between the values of $\Delta z_п$ and θ.

If $\Delta z_n < \theta$ for obtaining future values of the i−th coordinate is sufficient to find the value of optimal control, which corresponds to the trajectory of the SBA (Fig. 1). Then for value $z = \theta$ to get $x_i = x_i + \Delta x_i = f_i(x_i, ..., u, z)$ from (33) and put them in equation (23).

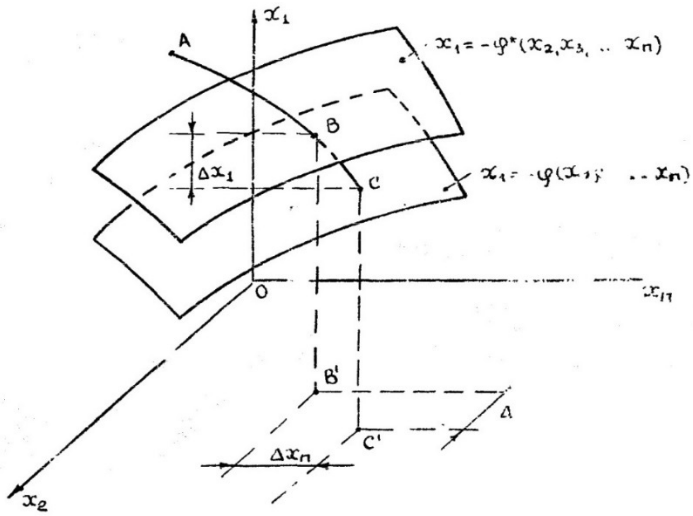

Fig.1 Geometric interpretation of Bess's method

If $\Delta z_n \geq \theta$ it is necessary to put $z = \theta - z_n$ in the equation (24). In this case the initial values $x_i^0 (i = \overline{1, n})$ lying on the new optimal switching surface, the next directly behind the original surface, and U is the new value of the control corresponding to the forced optimal motion after the new switching surface and founded x_i from (24) put in the equation (23).

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Our task corresponds to the first case. Then, in order to maintain a stable motion of the system under $t \geq t_k$ the condition of the equilibrium of the control system, it will be determined as $u(t) = 0$ on the finite interval $[t_k - \theta, t_k]$.

In fact, in our case, this control disconnection surface is a tube inside which there is a software path

Conclusions

The modal synthesis of linear closed loop stationary systems with transport delay and with the optimal control law (11) in accordance with the shown procedure can provide the required dynamic properties in them as proposed in the article. The procedure for modal synthesis of an optimal control law is carried outbased on the undefined coefficients method proposed in this article.Delay compensation method [8] is proposed to use in order to eliminate the stable self-oscillations occurrence (due to the delay) at the stabilization process end point near given trajectory.

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